

Electrodynamics of Bose-Einstein condensates in angular motion

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Abstract

A theory determining the electric and magnetic properties of vortex states in Bose-Einstein condensates (BECs) is presented. The principal ingredient is the Lagrangian of the system which we derive correct to the first order in the atomic centre of mass velocity. For the first time using centre of mass coordinates, a gauge transformation is performed and relevant relativistic corrections are included. The Lagrangian is symmetric in the electric and magnetic aspects of the problem and includes two key interaction terms, namely the Aharonov-Casher and the Röntgen interaction terms. The constitutive relations, which link the electromagnetic fields to the matter fields via their electric polarisation and magnetisation, follow from the Lagrangian as well as the corresponding Hamiltonian. These relations, together with a generalised Gross-Pitaevskii equation, determine the magnetic (electric) monopole charge distributions accompanying an order n vortex state when the constituent atoms are characterised by an electric dipole (magnetic dipole). Field distributions associated with electric dipole active (magnetic dipole active) BECs in a vortex state are evaluated for an infinite- and a finite-length cylindrical BEC. The predicted monopole charge distributions, both electric and magnetic, automatically satisfy the requirement of global charge neutrality and the derivations highlight the exact symmetry between the electric and magnetic properties. Order of magnitude estimates of the effects are given for an atomic gas BEC, superfluid helium and a spin-polarised hydrogen BEC.

1 Introduction

Recent years have seen the emergence of the physics of cold atomic ensembles and Bose-Einstein condensates (BECs), thanks to advances in laser cooling and trapping [1]. Atomic ensembles can now be routinely cooled and confined and, ultimately, made to condense under various conditions [1-3], most notably in magnetic and light traps, near surfaces, in material cavities and in optical lattices. The trend is towards producing larger systems with increasing numbers of constituents and increasing densities than currently. Furthermore, continuous evaporative cooling in optically trapped bosonic and fermionic gases have been demonstrated in stable optical traps and the techniques promise the attainment of BECs with very high densities and much larger coherence lifetimes [4].

In parallel with these experimental advances, various theoretical investigations [5] have been made, with increasing recent emphasis directed towards the angular motion of BECs in the form of vortex states [6-9] and the electromagnetic features associated with them. In particular, Leonhardt and Piwnicki [7] examined the magnetic effects due to the angular motion of a BEC whose constituent atoms are characterised by electric dipoles, thus forming a quantum dielectric in angular motion. They predicted that such a motion should give rise to a magnetic monopole positioned at the core of an $n = 1$ vortex. More recently, Shevshenko considered the converse case and put forward a theory for the generation of vortices in superfluid films by application of crossed electric and magnetic fields [9].

The situation is now ripe for a rigorous theory of the electromagnetic properties of BECs in rotational motion to be constructed. In contemplating such a theory, an immediate question which springs to mind is as follows. Hitherto, only the electromagnetic properties of dielectric BECs, namely those whose constituent atoms are characterised by an electric dipole moment (electric BECs), have been considered. What if the BEC atoms are characterised by a magnetic dipole moment (magnetic BECs)?

This question is now of practical relevance since BECs can be created in light traps, allowing, the magnetic properties of, for instance, alkali metal BECs to be investigated [8]. More significantly, explicitly magnetic BECs, particularly spin-polarised hydrogen BECs, have been created [10]. Intuitively one expects the physics of the problem to display a symmetry between the electric and magnetic aspects. Thus if a rotating electric BEC generates a magnetic monopole, then a rotating magnetic BEC should generate an electric monopole. One of the main tasks of the theory presented here is to test the validity of such an expectation.

In this paper we show that rotating magnetic BECs are indeed predicted to give rise to electric monopoles and that it is possible to formulate a theory in which both types of effect arise in a unified manner. However, our theory predicts that rotating BECs should not, as in Ref.[7] give rise to localised monopoles, but, rather, to monopole distributions. The treatment automatically ensures that charge neutrality is correctly preserved by the electric and magnetic monopole distributions associated with the vortex states of the two types of BEC.

The outline of this paper is as follows. In section 2 we begin by introducing the model system as a single atom interacting with the electromagnetic potentials via the familiar charge and current sources. We then systematise the procedure based on a gauge transformation leading from the conventional Lagrangian of this system to a form displaying the convective velocity-dependent interactions. These interactions involve the coupling of the electric and magnetic field intensities to the entire series of electric and magnetic multipoles in exact closed forms. In particular, we identify the Röntgen interaction [11] and we make use of symmetry arguments to incorporate an Aharonov-Casher interaction term [12]. In section 3 we derive the field equations which permit the identification of constitutive relations connecting matter fields to electromagnetic fields. We also derive the single-atom plus fields Hamiltonian and make use of this in section 4 to construct the many-body formulation appropriate for an ensemble of such atoms forming a BEC. A main outcome of the many-body theory is a systematic

derivation of a generalised Gross-Pitaevskii equation. In section 5 we consider rotating BECs and solve the constitutive relations, together with the Gross-Pitaevskii equation, to determine distributions for an order n vortex state of the two types of BEC. We also display the results for typical situations involving order of magnitude estimates of the effects, evaluated for an atomic gas BEC, superfluid helium and spin-polarised hydrogen BEC. Section 6 concludes with a summary and provides further comments.

2 Lagrangian

It suffices to model the constituent BEC atom in terms of a neutral two-particle system of mass $M = m_1 + m_2$ involving two bound charges $e_1 = -e_2 = e$ of masses m_1 and m_2 , position vectors \mathbf{q}_i and velocities $\dot{\mathbf{q}}_i$ where $i = 1, 2$. The conventional Lagrangian of this two-particle system coupled to the electromagnetic scalar potential ϕ and vector potential \mathbf{A} in an arbitrary gauge is written in the following form, which is non-relativistic in the motion of both particles [14-17],

$$L = \frac{1}{2}m_1\dot{\mathbf{q}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{q}}_2^2 + \int d^3\mathbf{r} \left[\frac{1}{2}\epsilon_0 \left(\mathbf{E}^2(\mathbf{r}) - c^2\mathbf{B}^2(\mathbf{r}) \right) + \mathbf{J}(\mathbf{r}) \cdot \mathbf{A}(\mathbf{r}) - \rho(\mathbf{r})\phi(\mathbf{r}) \right], \quad (1)$$

where $\mathbf{E} = -\dot{\mathbf{A}} - \nabla\phi$ and $\mathbf{B} = \nabla \times \mathbf{A}$. The charge and current densities are those appropriate for point charges, namely

$$\rho(\mathbf{r}) = \sum_{i=1,2} e_i\delta(\mathbf{r} - \mathbf{q}_i); \quad \mathbf{J}(\mathbf{r}) = \sum_{i=1,2} e_i\dot{\mathbf{q}}_i\delta(\mathbf{r} - \mathbf{q}_i) \quad (2)$$

In order to reveal the bound nature of the two particles and exhibit the desired effects involving gross motion of the whole atom we need to express the theory in terms of the centre of mass and relative coordinates. We therefore introduce the centre of mass coordinate \mathbf{R} and internal coordinate \mathbf{q} by [16]

$$\mathbf{R} = \frac{(m_1\mathbf{q}_1 + m_2\mathbf{q}_2)}{M}; \quad \mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2 \quad (3)$$

This step paves the way for the application of a Power-Zienau-Woolley (PZW) gauge transformation [16], to be carried out next at the Lagrangian level. As far as the authors

are aware, this is the first time this method has been performed when using centre of mass coordinates, although similar results were obtained using a multi-polar expansion on the equivalent Hamiltonian [14]. The generator of the gauge transformation is defined by

$$\chi(\mathbf{r}) = \frac{1}{e} \int d^3\mathbf{r}' \mathbf{A}(\mathbf{r}') \cdot \mathcal{F}_m(\mathbf{r}, \mathbf{r}') \quad (4)$$

where $\mathcal{F}_m(\mathbf{r}, \mathbf{r}')$ is a gauge vector function given by

$$\mathcal{F}_m(\mathbf{r}, \mathbf{r}') = e \int_0^1 d\lambda (\mathbf{r} - \mathbf{R}) \delta(\mathbf{r}' - \mathbf{R} - \lambda(\mathbf{r} - \mathbf{R})) \quad (5)$$

Note that despite superficial appearances, this vector function is not the electric multipolar polarisation field in closed form. The gauge transformation involves replacing the vector potential \mathbf{A} and scalar potential ϕ in the Lagrangian Eq.(1) by \mathbf{A}' and ϕ' related to the untransformed potentials by

$$\mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) - \nabla\chi(\mathbf{r}); \quad \phi'(\mathbf{r}) = \phi(\mathbf{r}) + \frac{\partial\chi(\mathbf{r})}{\partial t} \quad (6)$$

The transformed Lagrangian L' is

$$L' = L - \int d^3\mathbf{r} \left\{ \mathbf{J}(\mathbf{r}) \cdot \nabla\chi(\mathbf{r}) + \rho(\mathbf{r}) \frac{\partial\chi(\mathbf{r})}{\partial t} \right\} \quad (7)$$

It is straightforward, albeit laborious, to show that the terms involving the gradient and time derivative of the gauge function are given by

$$\nabla\chi = \mathbf{A}(\mathbf{r}) + \frac{1}{e} \int d^3\mathbf{r}' \mathbf{\Theta}_m(\mathbf{r}, \mathbf{r}') \times \mathbf{B}(\mathbf{r}') \quad (8)$$

and

$$\begin{aligned} \frac{\partial\chi}{\partial t} &= \frac{1}{e} \int d^3\mathbf{r}' \left\{ \dot{\mathbf{A}}(\mathbf{r}') \cdot \mathcal{F}_m(\mathbf{r}, \mathbf{r}') + \left(\dot{\mathbf{R}} \cdot \frac{\partial}{\partial \mathbf{R}} \right) [\mathbf{A}(\mathbf{r}') \cdot \mathcal{F}_m(\mathbf{r}, \mathbf{r}')] \right\} \\ &= -\dot{\mathbf{R}} \cdot \mathbf{A}(\mathbf{R}) + \frac{1}{e} \int d^3\mathbf{r}' \left\{ \dot{\mathbf{A}}(\mathbf{r}') \cdot \mathcal{F}_m(\mathbf{r}, \mathbf{r}') + \dot{\mathbf{R}} \cdot [(\mathcal{F}_m(\mathbf{r}, \mathbf{r}') - \mathbf{\Theta}_m(\mathbf{r}, \mathbf{r}')) \times \mathbf{B}(\mathbf{r}')] \right\} \end{aligned} \quad (9)$$

with

$$\mathbf{\Theta}_m(\mathbf{r}, \mathbf{r}') = e \int_0^1 d\lambda \lambda (\mathbf{r} - \mathbf{R}) \delta(\mathbf{r}' - \mathbf{R} - \lambda(\mathbf{r} - \mathbf{R})) \quad (10)$$

Substituting for \mathbf{J} and ρ using Eq.(2) and expressing the coordinates \mathbf{q}_1 and \mathbf{q}_2 in terms of \mathbf{R} and \mathbf{q} using Eq.(3) we have

$$L' = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\bar{m}\dot{q}^2 + \int d^3\mathbf{r} \left[\frac{1}{2}\epsilon_0 \{ \mathbf{E}^2(\mathbf{r}) - c^2 \mathbf{B}^2(\mathbf{r}) \} + \mathcal{P}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) + \mathcal{M}(\mathbf{r}) \cdot \mathbf{B}(\mathbf{r}) - \dot{\mathbf{R}} \cdot \{ \mathcal{P}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) \} \right], \quad (11)$$

where $\bar{m} = m_1 m_2 / M$ is the reduced mass. The vector fields $\mathcal{P}(\mathbf{r})$ and $\mathcal{M}(\mathbf{r})$ are, respectively, the electric polarisation and magnetisation associated with the charge and current sources of the atom, both of which emerge here as closed integral forms representing the entire electric and magnetic multipole series to all orders

$$\mathcal{P}(\mathbf{r}) = \sum_{i=1,2} \frac{e_i}{e} \mathcal{F}(\mathbf{q}_i, \mathbf{r}) \quad (12)$$

$$\mathcal{M}(\mathbf{r}) = \sum_{i=1,2} \frac{e_i}{e} \mathbf{\Theta}(\mathbf{q}_i, \mathbf{r}) \times (\dot{\mathbf{q}}_i - \dot{\mathbf{R}}) = \left[\frac{m_2}{M} \mathbf{\Theta}(\mathbf{q}_1, \mathbf{r}) + \frac{m_1}{M} \mathbf{\Theta}(\mathbf{q}_2, \mathbf{r}) \right] \times \dot{\mathbf{q}} \quad (13)$$

Note that the magnetisation field vector of the two-particle system involves only orbital magnetic multipoles, rather than the spin magnetic moment of the particles, but it is possible to generalise the theory to incorporate spin.

The interpretation of each of the various terms in the gauge-transformed Lagrangian in Eq.(11) is straightforward, except for last term, identified as the Röntgen interaction term [7,11], which involves the coupling between the centre of mass motion, the magnetic flux density and the electric polarisation field of the system.

The presence of the Röntgen interaction term makes the Lagrangian unsymmetric between the electric and magnetic properties, specifically as far as the centre of mass motion is concerned. This lack of symmetry suggests that an interaction term is missing which, like the Röntgen term, should be first order in $\dot{\mathbf{R}}$ and which couples the magnetisation to the electric field via the centre of mass motion. The missing term is identified as the Aharonov-Casher term [12] and is obtainable by adding a relativistic correction to the Lagrangian in Eq.(1) so that $\rho \rightarrow \rho'$ where

$$\rho'(\mathbf{r}) = \rho(\mathbf{r}) + \frac{1}{c^2} \dot{\mathbf{R}} \cdot \{ \nabla \times \mathcal{M}(\mathbf{r}) \}. \quad (14)$$

Including this correction in the original Lagrangian, and following the same procedures as descibed above, leads to the appearance of a new term in the form of the Aharonov-Casher term so that the new Lagrangian should be

$$L' = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\overline{m}\dot{\mathbf{q}}^2 + \int d^3\mathbf{r} \left[\frac{1}{2}\epsilon_0 \left(\mathbf{E}^2(\mathbf{r}) - c^2\mathbf{B}^2(\mathbf{r}) \right) + \mathcal{P}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) \right. \\ \left. + \mathcal{M}(\mathbf{r}) \cdot \mathbf{B}(\mathbf{r}) - \dot{\mathbf{R}} \cdot \left\{ \mathcal{P}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) - \frac{1}{c^2} \mathcal{M}(\mathbf{r}) \times \mathbf{E}(\mathbf{r}) \right\} \right], \quad (15)$$

which is seen to be symmetric between electric and magnetic interactions. Note, however, that the Röntgen term was obtained without introducing relativity while the Aharonov-Casher term required a relativistic correction.

3 Single atom plus fields system

3.1 Canonical momenta

The canonical variables in the transformed Lagrangian are the centre of mass coordinate \mathbf{R} and internal coordinate \mathbf{q} for the particle system and the vector and scalar potentials \mathbf{A} and ϕ for the fields. The corresponding canonical momenta are as follows. For the centre-of-mass motion we have

$$\mathbf{P} = \frac{\partial L'}{\partial \dot{\mathbf{R}}} = M\dot{\mathbf{R}} - \int d^3\mathbf{r} \left\{ \mathcal{P}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) - \frac{1}{c^2} \mathcal{M}(\mathbf{r}) \times \mathbf{E}(\mathbf{r}) \right\} \quad (16)$$

while for the internal motion we have

$$\mathbf{p} = \frac{\partial L'}{\partial \dot{\mathbf{q}}} = \overline{m}\dot{\mathbf{q}} - \left[\frac{m_2}{M} \boldsymbol{\Theta}(\mathbf{q}_1, \mathbf{r}) + \frac{m_1}{M} \boldsymbol{\Theta}(\mathbf{q}_2, \mathbf{r}) \right] \times \left[\mathbf{B}(\mathbf{r}) - \frac{1}{c^2} \dot{\mathbf{R}} \times \mathbf{E}(\mathbf{r}) \right] \quad (17)$$

For the fields we obtain the canonical momentum corresponding to \mathbf{A} as

$$\boldsymbol{\Pi} = \frac{\partial \mathcal{L}'}{\partial \dot{\mathbf{A}}} = -\epsilon_0 \mathbf{E} - \mathcal{P} - \frac{1}{c^2} \dot{\mathbf{R}} \times \mathcal{M} \quad (18)$$

where \mathcal{L}' is the Lagrangian density, identified in L' as the integrand in Eq. (15). The canonical momentum corresponding to ϕ vanishes.

3.2 Constitutive relations

The field equations follow from the Euler-Lagrange equations using ϕ and \mathbf{A} as canonical variables. They are identified as the familiar Maxwell equations

$$\nabla \cdot \mathbf{D} = 0; \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \quad (19)$$

where \mathbf{D} is the electric displacement field and \mathbf{H} is the magnetic field intensity which enter Maxwell's equations Eq.(19) provided that the following constitutive relations hold

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0} \mathbf{B}(\mathbf{r}) - \mathcal{M} + \dot{\mathbf{R}} \times \mathcal{P} \quad (20)$$

$$\mathbf{D}(\mathbf{r}) = \epsilon_0 \mathbf{E}(\mathbf{r}) + \mathcal{P} + \frac{1}{c^2} \dot{\mathbf{R}} \times \mathcal{M} \quad (21)$$

It is seen that these relations are themselves symmetric between the electric and magnetic contributions and we note, in particular, that the last term in Eq.(21) arises directly from the Aharonov-Casher Lagrangian interaction term. The corresponding term in Eq.(20) arises from the Röntgen Lagrangian interaction term.

3.3 Hamiltonian

The Hamiltonian of the atomic system interacting with the electromagnetic fields now follows from the Lagrangian by use of the canonical prescription

$$\mathcal{H} = \mathbf{P} \cdot \dot{\mathbf{R}} + \mathbf{p} \cdot \dot{\mathbf{q}} + \mathbf{\Pi} \cdot \dot{\mathbf{A}} - L' \quad (22)$$

which yields, after some manipulations,

$$\mathcal{H} = \frac{P^2}{2M} + \frac{p^2}{2m} + \frac{1}{2}\epsilon_0 \int d^3\mathbf{r} \left\{ \frac{1}{\epsilon_0^2} \mathbf{\Pi}^2(\mathbf{r}) + c^2 \mathbf{B}^2(\mathbf{r}) \right\} + \mathcal{H}_{int} \quad (23)$$

where \mathcal{H}_{int} accounts for the coupling between the electromagnetic fields and the atomic system (ignoring diamagnetic terms)

$$\mathcal{H}_{int} = \int d^3\mathbf{r} \left(\frac{1}{2\epsilon_0} \{ \mathcal{P}(\mathbf{r}) \}^2 + \frac{1}{\epsilon_0} \mathcal{P}(\mathbf{r}) \cdot \mathbf{\Pi}(\mathbf{r}) - \mathcal{M}(\mathbf{r}) \cdot \mathbf{B}(\mathbf{r}) \right)$$

$$\begin{aligned}
& + \frac{1}{2M} \int d^3\mathbf{r} [\mathbf{P} \cdot \{\mathcal{P}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) + \mu_0 \mathcal{M}(\mathbf{r}) \times \mathbf{\Pi}(\mathbf{r})\} \\
& + \{\mathcal{P}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) + \mu_0 \mathcal{M}(\mathbf{r}) \times \mathbf{\Pi}(\mathbf{r})\} \cdot \mathbf{P}]
\end{aligned} \tag{24}$$

where the magnetisation field \mathcal{M} should now be symmetrised in terms of the internal coordinate and its canonical momentum. Also in Eq.(24), pending further clarification, the term involving the square of the polarisation is deliberately included in the interaction Hamiltonian, although part of this term accounts for the Coulomb interaction between the particles.

4 Many-body formalism

4.1 Quantum field theoretic Hamiltonian

The single atom plus fields Hamiltonian now needs to be generalised to the many-body situation involving an ensemble of atoms plus fields interacting quantum-mechanically. We begin by introducing the boson field operator $\hat{\Psi}$ describing the internal as well as the centre-of mass motion of the ensemble, written as a sum over product eigenstates as follows

$$\hat{\Psi}(\mathbf{q}, \mathbf{R}) = \hat{\psi}(\mathbf{R}) \hat{\chi}(\mathbf{q}) = \sum_{n_1, n_2} \psi_{n_1}(\mathbf{R}) \chi_{n_2}(\mathbf{q}) \hat{a}_{n_1, n_2} \tag{25}$$

where $\psi_{n_1}(\mathbf{R})$ are state functions associated with the centre of mass and $\chi_{n_2}(\mathbf{q})$ with the internal atomic motion. The labels n_1 and n_2 incorporate all quantum numbers specifying these states. The operators \hat{a}_{n_1, n_2} and $\hat{a}_{n_1, n_2}^\dagger$ are annihilation and creation operators satisfying bosonic commutation relations. Completeness demands that we must have

$$\sum_{n_1} \psi_{n_1}^*(\mathbf{R}) \psi_{n_1}(\mathbf{R}') = \delta(\mathbf{R} - \mathbf{R}'); \quad \sum_{n_2} \chi_{n_2}^*(\mathbf{q}) \chi_{n_2}(\mathbf{q}') = \delta(\mathbf{q} - \mathbf{q}') \tag{26}$$

In the next step the many-body Hamiltonian operator is written as follows

$$\hat{H} = \int d^3\mathbf{q} d^3\mathbf{R} \hat{\Psi}^\dagger(\mathbf{q}, \mathbf{R}) \left(\frac{P^2}{2M} + \frac{p^2}{2\bar{m}} \right) \hat{\Psi}(\mathbf{q}, \mathbf{R}) + \frac{1}{2}\epsilon_0 \int d^3\mathbf{r} \left\{ \frac{1}{\epsilon_0^2} \mathbf{P}^2(\mathbf{r}) + c^2 \mathbf{B}^2(\mathbf{r}) \right\}$$

$$\begin{aligned}
& + \int d^3\mathbf{q}d^3\mathbf{R}\hat{\Psi}^\dagger(\mathbf{q},\mathbf{R})V_{trap}\hat{\Psi}(\mathbf{q},\mathbf{R}) \\
& + \frac{1}{2}\int d^3\mathbf{q}d^3\mathbf{R}\int d^3\mathbf{q}'d^3\mathbf{R}'\hat{\Psi}^\dagger(\mathbf{q},\mathbf{R})\hat{\Psi}^\dagger(\mathbf{q}',\mathbf{R}')U(\mathbf{R}-\mathbf{R}')\hat{\Psi}(\mathbf{q},\mathbf{R})\hat{\Psi}(\mathbf{q}',\mathbf{R}') \\
& + \mathcal{H}_{int}^Q
\end{aligned} \tag{27}$$

where \mathcal{H}_{int}^Q is given by

$$\begin{aligned}
\mathcal{H}_{int}^Q &= \int d^3\mathbf{r} \left[\frac{1}{2\epsilon_0} \{\hat{\mathcal{P}}(\mathbf{r})\}^2 + \frac{1}{\epsilon_0} \hat{\mathcal{P}}(\mathbf{r}) \cdot \mathbf{\Pi}(\mathbf{r}) - \hat{\mathcal{M}}(\mathbf{r}) \cdot \mathbf{B}(\mathbf{r}) \right] \\
& + \frac{1}{2M} \int d^3\mathbf{q}d^3\mathbf{R} \int d^3\mathbf{r} \hat{\Psi}^\dagger(\mathbf{q},\mathbf{R}) \left[\mathbf{P} \cdot \left\{ \hat{\mathcal{P}}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) + \mu_0 \hat{\mathcal{M}}(\mathbf{r}) \times \mathbf{\Pi}(\mathbf{r}) \right\} \right. \\
& + \left. \left\{ \hat{\mathcal{P}}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) + \mu_0 \hat{\mathcal{M}}(\mathbf{r}) \times \mathbf{\Pi}(\mathbf{r}) \right\} \cdot \mathbf{P} \right] \hat{\Psi}(\mathbf{q},\mathbf{R})
\end{aligned} \tag{28}$$

with the polarisation and magnetisation converted to field theoretic operators as follows:

$$\hat{\mathcal{P}} = \sum_{i=1,2} e_i \int d^3\mathbf{q}d^3\mathbf{R} \hat{\Psi}^\dagger(\mathbf{q},\mathbf{R}) \int_0^1 d\lambda (\mathbf{q}_i - \mathbf{R}) \delta(\mathbf{r} - \mathbf{R} - \lambda(\mathbf{q}_i - \mathbf{R})) \hat{\Psi}(\mathbf{q},\mathbf{R}) \tag{29}$$

$$\hat{\mathcal{M}} = \sum_{i=1,2} \frac{1}{2m_i} \int d^3\mathbf{q}d^3\mathbf{R} \hat{\Psi}^\dagger(\mathbf{q},\mathbf{R}) [\mathbf{\Theta}(\mathbf{q}_i, \mathbf{r}) \times \mathbf{p} - \mathbf{p} \times \mathbf{\Theta}(\mathbf{q}_i, \mathbf{r})] \hat{\Psi}(\mathbf{q},\mathbf{R}) \tag{30}$$

Note that although the polarisation and magnetisation operators are expressed above in terms of \mathbf{q}_1 and \mathbf{q}_2 , they can readily be written in terms of the relative and centre-of-mass coordinates using Eq.(3).

We have introduced two new terms in the many-body Hamiltonian Eq.(27). The term involving V_{trap} accounts for the trapping potential used to confine the atomic ensemble, and the term involving $U(\mathbf{R} - \mathbf{R}')$ is identified as the hard sphere collision term. This is usually taken to be of the form

$$U(\mathbf{R} - \mathbf{R}') = \frac{2\pi\hbar^2 a}{M} \delta(\mathbf{R} - \mathbf{R}') \equiv U_0 \delta(\mathbf{R} - \mathbf{R}') \tag{31}$$

where a is the scattering length.

4.2 Generalised Gross-Pitaevskii equation

Having arrived at the appropriate many-body quantum field theoretical Hamiltonian we can now derive the Schrödinger equation satisfied by the atomic field $\hat{\Psi}(\mathbf{q}, \mathbf{R})$. This formally follows from the Heisenberg equation

$$i\hbar\dot{\hat{\Psi}} = [\hat{H}, \hat{\Psi}] = E\hat{\Psi} \quad (32)$$

It is instructive to check first what the outcome would be in the absence of all electromagnetic terms plus terms involving the polarisation. In this case Eq.(32) yields

$$\left[\frac{p^2}{2M} + \frac{p^2}{2\bar{m}} + V_{trap} + U_0\hat{\Psi}^\dagger(\mathbf{q}, \mathbf{R})\hat{\Psi}(\mathbf{q}, \mathbf{R}) \right] \hat{\Psi}(\mathbf{q}, \mathbf{R}) = E\hat{\Psi}(\mathbf{q}, \mathbf{R}) \quad (33)$$

Apart from the implicit dependence on the internal quantum states $\chi(\mathbf{q})$ and the explicit appearance of the kinetic energy term $p^2/2\bar{m}$, Eq.(33) is reminiscent of the Gross-Pitaevskii equation prior to assuming that all ensemble constituents occupy the ground quantum state.

On reintroducing the electromagnetic interaction terms the full Schrödinger equation obtained turns out to be a modified Goss-Pitaevskii equation. The modifications involve the presence of the internal states, including their coupling to the gross motion and the coupling of the two motions to the electromagnetic fields. In order to derive this general equation we need to discuss the origins of various terms which will appear in it. First there must be a term accounting for the binding of the particles within a single atom, i.e. intra-atom Coulomb interactions, and there must also be terms accounting for the interaction between the atoms, i.e. the inter-atom interactions. Both Coulmb interactions arise from the field theoretic term

$$\hat{H}_c = \frac{1}{2\epsilon_0} \int d^3\mathbf{r} \left([\hat{\mathcal{P}}_{intra}^\parallel]^2 + [\hat{\mathcal{P}}_{inter}]^2 \right) \quad (34)$$

We have dropped the additional term involving the transverse part of the intra-atom polarisation, as this can be shown to lead to infinite self energies. The full inter-atom polarisation term gives the electromagnetic interactions between the atoms.

Assuming that inter-atomic separations $|\mathbf{R} - \mathbf{R}'|$ are typically much larger than a dipole length $|\mathbf{q}|$, which amounts to the dipole approximation, we can write

$$\mathcal{P} = N\mathbf{d}|\psi(\mathbf{R})|^2; \quad \mathcal{M} = N\boldsymbol{\mu}|\psi(\mathbf{R})|^2 \quad (35)$$

where N is the number of atoms in the ensemble, \mathbf{d} is the electric dipole moment and $\boldsymbol{\mu}$ is the magnetic dipole moment. The modified Gross-Pitaevskii equation in the dipole approximation turns out to be in the form

$$\left[\frac{P^2}{2M} + V_{trap} + \frac{p^2}{2\bar{m}} - \frac{e^2}{4\pi\epsilon_0 q} + U_0 \hat{\Psi}^\dagger(\mathbf{q}, \mathbf{R}) \hat{\Psi}(\mathbf{q}, \mathbf{R}) + V_{dd} + V_{em} \right] \hat{\Psi}(\mathbf{q}, \mathbf{R}) = E \hat{\Psi}(\mathbf{q}, \mathbf{R}) \quad (36)$$

where V_{dd} is an inter-atom interaction term

$$V_{dd} = \frac{e}{\epsilon_0} \int d^3\mathbf{q}' (\mathbf{d} \cdot \mathbf{q}') \left\{ 2\hat{\Psi}^\dagger(\mathbf{q}', \mathbf{R}) \hat{\Psi}(\mathbf{q}', \mathbf{R}) - \hat{\Psi}^\dagger(\mathbf{q}', \mathbf{R}) \hat{\Psi}(\mathbf{q}', \mathbf{R}) \right\} \quad (37)$$

while V_{em} includes the coupling of the electric and magnetic dipole systems, pertaining to the internal dynamics, with the centre of mass and the electromagnetic fields

$$V_{em} = -\mathbf{d} \cdot \mathbf{E}^\perp(\mathbf{R}) - \boldsymbol{\mu} \cdot \mathbf{B}(\mathbf{R}) + \frac{1}{2M} \left[\mathbf{P} \cdot \left\{ \mathbf{d} \times \mathbf{B}(\mathbf{R}) - \frac{1}{c^2} \boldsymbol{\mu} \times \mathbf{E}(\mathbf{R}) \right\} + \left\{ \mathbf{d} \times \mathbf{B}(\mathbf{R}) - \frac{1}{c^2} \boldsymbol{\mu} \times \mathbf{E}(\mathbf{R}) \right\} \cdot \mathbf{P} \right] \quad (38)$$

Next we assume that the number of atoms in the BEC is very large and that they all occupy the ground state. Under such circumstances only the ground state operators are involved and these become c-numbers such that

$$\hat{a}_{0,0} \approx \sqrt{N}; \quad \hat{a}_{0,0}^\dagger \approx \sqrt{N} \quad (39)$$

All terms involving other operators \hat{a}_{n_1, n_2} and $\hat{a}_{n_1, n_2}^\dagger$ where $n_1 \neq 0$ and $n_2 \neq 0$ give vanishing results. Although this greatly simplifies the problem, the internal and the centre-of-mass motions are still coupled. To achieve a decoupling of the two motions we need to perform ensemble averaging. This allows us to write

$$V_{dd} = \frac{e}{\epsilon_0} N |\psi_0(\mathbf{R})|^2 \int d^3\mathbf{q}' \mathbf{d} \cdot \mathbf{q}' |\chi_0(\mathbf{q}')|^2 \quad (40)$$

The ensemble averaging also results in the substitution

$$E = E_\mu + \mathcal{E}_0 \quad (41)$$

where E_μ is the chemical potential and \mathcal{E}_0 is the lowest internal energy, i.e. the ground state energy eigenvalue of Schrödinger's equation for the internal motion

$$\left\{ \frac{p^2}{2\bar{m}} - \frac{e^2}{4\pi\epsilon_0 q} \right\} \chi_0(\mathbf{q}) = \mathcal{E}_0 \chi_0(\mathbf{q}) \quad (42)$$

with $\chi_0(\mathbf{q})$ the corresponding hydrogenic ground state eigenfunction. The final step is to assume that in a condensate there should be a maximum correlation between the atoms, in which case we can identify V_{dd} as an effective dipole-dipole interaction in the form

$$V_{dd} = \frac{d^2}{\epsilon_0} N |\psi_0(\mathbf{R})|^2 \quad (43)$$

The modified Gross-Pitaevskii equation now becomes

$$\left[\frac{P^2}{2M} + V_{trap} + N |\psi_0(\mathbf{R})|^2 \left(\frac{d^2}{\epsilon_0} + U_0 \right) + V_{em} \right] \hat{\Psi}(\mathbf{q}, \mathbf{R}) = E_\mu \hat{\Psi}(\mathbf{q}, \mathbf{R}) \quad (44)$$

Like the Gross-Pitaevskii equation in the absence of electromagnetic effects, this is a non-linear Schrödinger equation. The additional terms are V_{dd} , the dipole-dipole interaction, and V_{em} , the electromagnetic interaction. All the new terms have been arrived at systematically within our theory, including the dipole-dipole term which in previous considerations has been added phenomenologically. To our knowledge the new interaction terms included in V_{em} have not been derived rigorously before. To see the effects of the electromagnetic fields on the properties of a BEC we now consider the special case of a BEC in a vortex state

5 Rotating BECs

In view of the typical experimental arrangements for generating BECs we may assume that the BEC is confined within a cylindrical region $|z| < z_0$ with cylinder radius R_0 and height $2z_0$. The length z_0 can be larger or smaller than R_0 depending on the type of

trap, but below we shall consider the cases where $z_0 \approx R_0$ and z_0 infinite. We perform a Madelung transformation [6] and, assuming that the wavefunction is independent of z , write

$$\psi(\mathbf{R}) = \left(\frac{1}{4\pi z_0 R_0^2} \right)^{1/2} |\psi(r)| e^{in\phi} \quad (45)$$

where (r, ϕ, z) are the cylindrical coordinates of \mathbf{R} . On considering solutions of Eq. (44) we may drop the electromagnetic field interaction term, which introduces small second order corrections [7], and the dipole-dipole interaction term which is also negligible compared to the s-wave scattering term. We thus have for the velocity profile of the BEC in an order n vortex state

$$\dot{\mathbf{R}} = \left(\frac{n\hbar}{Mr} \right) \hat{\phi} \quad (46)$$

and also the radial wavefunction satisfies the equation [7]

$$\left[\frac{d^2}{d\xi^2} + \frac{1}{\xi} \frac{d}{d\xi} - \frac{n^2}{\xi^2} + \frac{2ME}{\hbar^2} - 4n_{1d}a|\psi(\xi)|^2 \right] \psi(\xi) = 0, \quad (47)$$

with a the scattering length first encountered in Eq.(31); $\xi = r/R_0$, $n_{1d} = N/2z_0$ is the one-dimensional atomic density parameter and we set $\psi(\xi = 1) = 0$. We note that the solution to this equation for $n_{1d}a = 0$ would be the Bessel function of the first kind, $J_n(x)$, with a root at $\xi = 1$. Figure 1(a) displays the modulus squared of the numerical solution of Eq.(47) for various values of $n_{1d}a$.

5.1 Electric BEC

In an electric BEC the constituent atoms are characterised by an electric dipole moment. We assume that the moment vectors all point in the z -direction. We concentrate on the static case and, setting $\mathbf{M} = 0$, we take the divergence of both sides in Eq. (20). The divergence of the magnetic flux density \mathbf{B} is automatically zero. Hence,

$$\begin{aligned} \nabla \cdot \mathbf{H} = \nabla \cdot (\dot{\mathbf{R}} \times \mathcal{P}) &= \mathcal{P} \cdot (\nabla \times \dot{\mathbf{R}}) - \dot{\mathbf{R}} \cdot (\nabla \times \mathcal{P}) \\ &= \frac{\hbar n}{Mr} \left[\delta(r) + \frac{d}{dr} \right] \frac{n_{1d}d}{2\pi R_0^2} |\psi(r)|^2 = \frac{\rho_m(r)}{\mu_0}. \end{aligned} \quad (48)$$

where we have expressed the right hand side in the first equality of Eq. (48) as a magnetic monopole charge density divided by μ_0 . The magnetic field intensity can be written as the gradient of a scalar field: $\mathbf{H} = -\nabla\Phi$, where Φ is the magnetic vector potential. This is similar to the result obtained in Ref.[7]. However, in [7] it was argued that, although the wavefunction vanishes on the cylinder axis, some atoms will leak into the core of the vortex and this gives rise to the monopole charge. The variations of the wavefunction across the cylinder were ignored. In solving Eq. (48) we note that the term involving the delta function in the first equality cancels with an identical term arising from integrating the derivative which is proportional to $|\psi(0)|^2$. This feature of our theory guarantees ‘charge neutrality’, in contrast with the case in Ref.[7].

Figure 1(b) displays the variation of the derivative of the modulus squared of the wavefunction. This quantity is proportional to the two-dimensional monopole charge density in a symmetry plane containing the cylinder axis and can be written in terms of $\xi = r/R_0$ as follows

$$2\pi\xi\rho_m(\xi) = \frac{\hbar n n_{1d} \mu_0 d}{M R_0^4} \left(\frac{d|\psi(\xi)|^2}{d\xi} \right) \quad (49)$$

Clearly as the scattering length becomes longer or the number of atoms increases the wavefunction broadens and the associated charges accumulate near the axis and edge of the cylinder. However, the integral of the density over the cylinder volume reduces to zero, as demanded by charge neutrality. This magnetic monopole charge distribution can be thought of as akin to that found in a cylindrical capacitor with one set of charges on an inner thin cylinder and another set of charges of opposite sign on an outer cylinder. A useful measure of the effect is the cumulative positive charge, Q_m , in the inner region of the cylinder, which can be estimated by integrating the charge density up to $r = R_0/2$ ($\xi = 1/2$). We find

$$Q_m = \frac{\hbar n N \mu_0 d}{M R_0^2} |\psi(\xi = 1/2)|^2 \quad (50)$$

For an infinite cylinder, Eq. (48) can be solved using Gauss’s law to obtain

$$\mathbf{H}(\xi) = \hat{\mathbf{r}} \frac{\hbar n n_{1d} d}{2\pi M R_0^3 \xi} \int_0^\xi \left[\delta(\xi') + \frac{d}{d\xi'} \right] |\psi(\xi')|^2 d\xi' = \hat{\mathbf{r}} \frac{\hbar n n_{1d} d}{2\pi M R_0^3 \xi} |\psi(\xi)|^2. \quad (51)$$

where $\hat{\mathbf{r}}$ is a cylindrical radial unit vector. We see that \mathbf{H} is proportional to $|\psi(\xi)|^2$ which vanishes outside the cylinder. There is a magnetic field only inside the cylinder, another signature of the conservation of ‘magnetic monopole charge’.

If we wish to detect the magnetetic effects associated with the vortex, it may be necessary to probe the region outside the BEC which can have a magnetic field for a cylinder of a finite height $2z_0$. The solution for the magnetic potential in the finite cylinder case can be derived using Green functions [18]. We find

$$\Phi(\xi, z) = \Phi_0^m \int_0^1 d\xi' \int_0^{2\pi} d\phi' \frac{\xi' - \xi \cos(\phi')}{h^2} \left[\frac{1 - z}{(h^2 f^2 + (1 - z)^2)^{1/2}} + \frac{1 + z}{(h^2 f^2 + (1 + z)^2)^{1/2}} \right], \quad (52)$$

where Φ_0^m is a scaling magnetic potential controlling the order of magnitude of the effect

$$\Phi_0^m = \frac{\hbar n n_{1d} d}{8\pi^2 M R_0^2} \quad (53)$$

the variable z is in units of z_0 and we have used the notation

$$h^2 = \xi^2 + \xi'^2 - 2\xi\xi' \cos(\phi'), \quad f = \frac{R_0}{z_0} \quad (54)$$

Figure 2 displays a contour plot of the magnetic potential on a symmetry plane through the cylinder axis and above the symmetry plane of $z = 0$. As expected, because of the finite height of the cylinder, the magnetic potential is not confined to the inside but leaks to the outside of the cylinder. Within the cylinder the magnetic potential exhibits its largest variations where the wavefunction changes most rapidly.

It is useful to obtain order of magnitude estimates of the effects just described for the cases of a typical atomic gas BEC and for superfluid helium. In the case of an atomic gas BEC, we choose typical parameters appropriate for ^{87}Rb [5] in the $n = 1$ vortex state, assuming that the BEC is confined in a cylinder of dimensions $R_0 = z_0 = 2\mu\text{m}$. Each atom is assumed to be characterised by a transition electric dipole moment $d = ea_B$, where $a_B = 0.53\text{\AA}$ is the Bohr radius, and the s-wave scattering length is taken to be $a = 59\text{\AA}$. Setting $n_{1d}a = 100$ gives a linear density of

$n_{1d} = 1.7 \times 10^{10} \text{m}^{-1}$ and a total number of atoms $N = 7 \times 10^4$. Thus we have for Φ_0^m

$$\Phi_0^m = 3.3 \times 10^{-19} \text{ A} \quad (55)$$

and for the positive magnetic monopole charge residing in the inner region of the cylinder, Eq. (50)

$$Q_m = 1.3 \times 10^{-28} |\psi(\xi = 1/2)|^2 \text{ Vs} \quad (56)$$

which is equivalent to a magnetic field of order 10^{-19}T .

In the case of superfluid helium, which is taken to be characterised by macroscopic electric susceptibility the polarisation field vector can be written as

$$\mathcal{P}(\xi) = \chi \mathbf{E} |\psi(\xi)|^2 \quad (57)$$

where \mathbf{E} is an applied electric field and $\chi = 0.052$ is the value of the susceptibility. We have, in effect replaced the volume density times the dipole moment with a susceptibility times an externally applied electric field. Thus, for every 1Vm^{-1} of applied field the magnetic monopole charge in the inner region of the cylinder is

$$Q_m = 1.2 \times 10^{-20} |\psi(\xi = 1/2)|^2 \text{ Vs} \quad (58)$$

and a magnetic field of order 10^{-14}T (equivalent to the values in [7]). This larger magnitude stems from the smaller helium mass and larger density (approximately 10^{21}m^{-3} in the case of the atomic BEC above and 10^{28}m^{-3} for superfluid helium). The potential estimated in Eq.(55) for the Rb atomic BEC is clearly too small to be measured on the basis of current experimental capabilities. However, one expects that denser atomic gas BECs will become available in the future. The corresponding estimate found above in the case of superfluid helium indicates that the effects are, indeed amenable to experimental detection, as deduced in [7].

5.2 Magnetic BEC

Next we consider to the case of a BEC in which the constituent atoms are characterised by magnetic dipoles aligned along the z -direction. Setting $\mathcal{P} = 0$, we take the

divergence of both sides in Eq. (21), and following the procedure in the electric BEC case, we obtain

$$\begin{aligned}\epsilon_0 \nabla \cdot \mathbf{E} &= -\frac{1}{c^2} \nabla \cdot (\dot{\mathbf{R}} \times \mathcal{M}) = -\frac{1}{c^2} [\mathcal{M}(\nabla \times \dot{\mathbf{R}}) - \dot{\mathbf{R}} \cdot (\nabla \times \mathcal{M})] \\ &= -\frac{\hbar n}{c^2 M r} \left[\delta(r) + \frac{d}{dr} \right] \frac{n_{1d} \mu}{2\pi R_0^2} |\psi(r)|^2 = \rho_e(r).\end{aligned}\quad (59)$$

It can be seen that, for the fields and potentials, Eq.(59) transforms to Eq.(48) when d is replaced by $-\mu_0\mu$. Therefore we can deduce the electric field associated with the vortex state of an infinitely long cylindrical BEC directly by simple substitution. The electrostatic potential is related to the electric field by , $\mathbf{E} = -\nabla\Phi$ and is given directly by by Eq.(52) and (53), with Φ_0^m replaced by

$$\Phi_0^e = -\frac{\hbar n n_{1d} \mu_0 \mu}{8\pi^2 M R_0^2} \quad (60)$$

Figure 1(b) now serves to display a function proportional to the areal electric charge density in a symmetry plane such that

$$2\pi\xi\rho_e(\xi) = -\frac{\hbar n n_{1d} \mu}{M c^2 R_0^4} \left(\frac{d|\psi(\xi)|^2}{d\xi} \right) \quad (61)$$

Figure 2 should now be taken to present a function proportional to the electrostatic potential. The corresponding cumulative negative electric charge residing in the inner region of the cylinder obtained by integrating ρ_e over the cylinder volume is approximately

$$Q_e = -\frac{\hbar n N \mu}{M c^2 R_0^2} |\psi(\xi = 1/2)|^2 \quad (62)$$

As to the order of magnitude, the best candidate for observing this effect is spin-polarised hydrogen [10] which has been produced as a BEC. The hydrogen is doubly spin-polarised, thus $\mu = 2\mu_b = \hbar e/m_e = 1.9 \times 10^{-23} \text{Am}^2$, a typical density is $5 \times 10^{21} \text{m}^{-3}$ and the s-wave scattering length is $a = 0.72 \text{\AA}$. Again assuming that $n_{1d}a = 100$ and $z_0 = 5 \text{mm}$ we obtain

$$\Phi_0^e = -2.7 \times 10^{-16} \text{ V} \quad \text{and} \quad Q_e = -1.9 \times 10^{-27} |\psi(\xi = 1/2)|^2 \text{C} \quad (63)$$

Clearly these magnitudes are too small for the effect to be experimentally detectable at present. However, the effect should become amenable to measurement whenever denser spin-polarised hydrogen BECs can be produced (with densities of the order of those of superfluid helium).

6 Comments and Conclusions

We have presented a rigorous treatment of the electromagnetic properties of BECs executing rotational motion when in an order n vortex state. The theory is based on a Lagrangian which we have constructed in such a manner as to emphasise the symmetry of the problem with regards to the centre of mass motion. The Lagrangian led us to constitutive relations which link the fields to the polarisation sources, including the convective contributions arising from the centre of mass motion. We have also shown how direct generalisation to the many-body situation can be done, leading to a modified Gross-Pitaevskii equation which incorporates familiar interactions plus the coupling to the electromagnetic fields. Applications of the theory to the case of a rotating BEC is straightforward and we have shown how this leads to the electric and magnetic properties of a rotating BEC, depending on whether the constituent atoms are electric dipole active or magnetic dipole active.

In particular, we have seen how a dipole-active BEC generates electric and magnetic monopole charge distributions and their associated electric and magnetic fields when the condensate is in a vortex state. We have also shown how, even if no condensate atoms occupy the vortex core, monopole charge distributions, both electric and magnetic, arise which are globally neutral. The problem becomes similar to that of finding the potential in a cylindrical capacitor. In the case of an atomic gas BEC of finite size of either kind, the vortex state generates fields outside it which are small when calculated on the basis of current estimates. Their experimental detection will have to await the advent of denser atomic gas BECs than are currently available. The predicted magnetic fields are much larger for a denser and more electrically polarisable system such as a superfluid. In the case of superfluid helium, we have shown that the magnetic distributions are sufficiently large to be amenable to experimental detection. It is envisaged that vortices generated in future denser BECs of spin-polarised hydrogen would give rise to electric distributions which could be measurable.

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Figure Captions

Figure 1

Variations of (a) the modulus squared of the normalised BEC wavefunction in the $n = 1$ vortex state and (b) its derivative with $n_{1d}a = 100$ (solid curve), $n_{1d}a = 10$ (dashed curve) and $n_{1d}a = 0.1$ (dotted curve).

Figure 2

A contour plot within a symmetry plane of the cylinder showing the variations of the electric (magnetic) potential of a magnetic (electric) dipole active BEC in the $n = 1$ vortex state which occupies the shaded region. The potential exhibits angular symmetry around $R = 0$, is symmetric about the plane $z = 0$ and is in units of Φ_0^e for the electric potential and Φ_0^m for the magnetic potential (see text). The condensate is such that $z_0/R_0 = 1$ and $n_{1d}a = 100$.